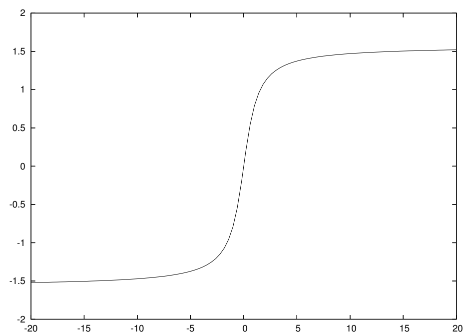
Name ______________________ Student Number ______________

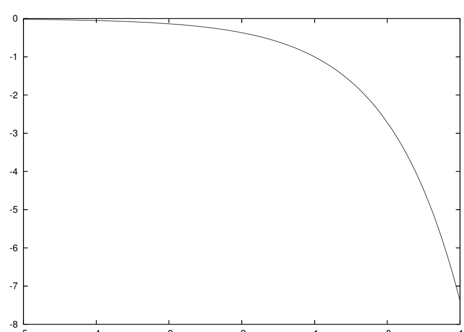
All solutions are to be presented on the paper in the space provided. The exam is closed book, no calculators. Time for the exam is 75 minutes.

(1) Draw the graphs of the following functions. Include the equations of any asymptotes and at least one known point.

(a) $\tan^{-1}(x)$



(b) $-e^{x+1}$



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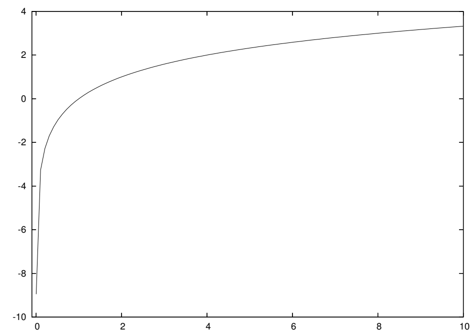
(c) $\log_2(x)$



(2) State the domain of the following functions:

(a) $\ln(x^2 - 1)$

$$\begin{array}{rcl} x^2 - 1 & > & 0 \\ x^2 & > & 1 \\ |x| & > & 1 \\ x < -1 & \text{or} & x > 1 \\ x \in (-\infty, -1) & \cup & (1, \infty) \end{array}$$

(b) $\dfrac{1}{1 - e^{2x}}$

$$\begin{array}{rcl} 1 - e^{2x} & = & 0 \\ e^{2x} & = & 1 \\ 2x & = & 0 \\ x & = & 0 \end{array}$$

(3) Solve the following equations:

(a) $\ln(x^2+1) = 4$

$$\begin{aligned} e^{\ln(x^2+1)} &= e^4 \\ x^2+1 &= e^4 \\ x^2 &= e^4 - 1 \\ x &= \pm(e^4-1) \end{aligned}$$

(b) $2^{x^{-1}} = 3$

$$\begin{aligned} \log_2(2^{x^{-1}}) &= \log_2 3 \\ x^{-1} &= \log_2 3 \\ x &= \frac{1}{\log_2 3} \end{aligned}$$

(4) Evaluate the following limits:

(a) $\displaystyle\lim_{x\to -1} \frac{x^2+3x+2}{x^2-x-2}$

$$\begin{aligned} \lim_{x\to -1} \frac{(x+1)(x+2)}{(x+1)(x-2)} &= \lim_{x\to -1} \frac{x+2}{x-2} \\ &= -\frac{1}{3} \end{aligned}$$

(b) $\displaystyle\lim_{x\to 4^+} \log_7(x-4)$

$x - 4 \to 0$ from the right, so $\displaystyle\lim_{x\to 4^+} \log_7(x-4) = -\infty$

(c) $\displaystyle\lim_{x\to -\infty} \frac{1}{3}e^x = 0$

(d) $\displaystyle\lim_{x\to \infty} (-x^5 + 2x^2 + 1) = -\infty$

(e) $\lim_{x\to 1} \frac{\sqrt{x^2+3}-2}{x-1}$

$$\begin{aligned}
\lim_{x\to 1} \frac{\sqrt{x^2+3}-2}{x-1} &= \lim_{x\to 1} \frac{\sqrt{x^2+3}-2}{x-1}\,\frac{\sqrt{x^2+3}+2}{\sqrt{x^2+3}+2} \\
&= \lim_{x\to 1} \frac{x^2+3-4}{(x-1)\sqrt{x^2+3}+2} \\
&= \lim_{x\to 1} \frac{x^2-1}{(x-1)\sqrt{x^2+3}+2} \\
&= \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)\sqrt{x^2+3}+2} \\
&= \lim_{x\to 1} \frac{x+1}{\sqrt{x^2+3}+2} \\
&= \frac{2}{4} \\
&= \frac{1}{2}
\end{aligned}$$

(f) $\lim_{x\to -2^-} \frac{|x+2|}{x+2}$

$$f(x) = \begin{cases} \frac{x+2}{x+2} & x+2 \geq 0 \\ \frac{-(x+2)}{x+2} & x+2 < 0 \end{cases} \quad = \quad \begin{cases} 1 & x \geq -2 \\ -1 & x < -2 \end{cases}$$

Since $x+2 \to 0$ from the left as $x \to -2$ from the left, the limit is -1.

(g) $\lim_{x\to 8} \left| \frac{1}{x-8} - \frac{1}{(x-8)^2} \right|$

$$\begin{aligned}
\lim_{x\to 8} \left| \frac{1}{x-8} - \frac{1}{(x-8)^2} \right| &= \lim_{x\to 8} \left| \frac{x-8-1}{(x-8)^2} \right| \\
&= \lim_{x\to 8} \left| \frac{x-9}{(x-8)^2} \right| \\
&= \infty
\end{aligned}$$

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(5) Evaluate the following limit and state which single key property was used: $\lim_{x\to 1} \sin(\pi x)$

$$\lim_{x\to 1} \sin(\pi x) = \sin(\pi) = 0$$

The first equality is because $\sin(x)$ is continuous.

(6) Answer the following questions for the given piecewise defined function:

$$f(x) = \begin{cases} \frac{x^2-3x-4}{x^2-6x-7} & x \neq -1 \\ 1 & x = -1 \end{cases}$$

(a) Show that $f(x)$ is not continuous at $x = -1$

$$\begin{aligned} \lim_{x\to -1} f(x) &= \lim_{x\to -1} \frac{x^2-3x-4}{x^2-6x-7} \\ &= \lim_{x\to -1} \frac{(x-4)(x+1)}{(x-7)(x+1)} \\ &= \lim_{x\to -1} \frac{x-4}{x-7} \\ &= \frac{-5}{-8} \\ &= \frac{5}{8} \end{aligned}$$

And $f(-1) = 1$. Since $\lim_{x\to -1} f(x) \neq f(-1)$, $f(x)$ is not continuous at $x = -1$.

(b) What type of discontinuity occurs at $x = -1$?
This is a jump discontinuity.

(c) Redefine $f(x)$ so that it is continuous everywhere.
Rewrite $f(x)$ as $f(x) = \dfrac{x-4}{x-7}$.

(7) Show that the following equation has at least one solution: $\sin(x-1) = -x$

Use the intermediate value theorem. Let $f(x) = \sin(x-1) + x$. This is a continuous function. Note that $f(1) = 1$ and $f(0) = \sin(-1) < 0$. Then, by the intermediate value theorem, there is a number, $c \in (0,1)$ such that $f(c) = 0$. This number c solves the initial equation.

(8) Evaluate the following limit: $\lim_{x\to 1} \left\{ (x-1)^2 \sin\left(\frac{\pi}{x^2-1}\right) \right\}$

This is a squeeze theorem problem.

$$-1 \leq \sin\left(\frac{\pi}{x^2-1}\right) \leq 1$$

$$-(x-1)^2 \leq (x-1)^2 \sin\left(\frac{\pi}{x^2-1}\right) \leq (x-1)^2$$

Since $\lim_{x\to 1}(-(x-1)^2) = \lim_{x\to 1}(x-1)^2 = 0$, the squeeze theorem implies that

$$\lim_{x\to 1} \left\{ (x-1)^2 \sin\left(\frac{\pi}{x^2-1}\right) \right\} = 0$$